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Supersymmetric Nonlinear Sigma Models

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Abstract

Supersymmetric nonlinear sigma models are formulated as gauge theories. Auxiliary chiral superfields are introduced to impose supersymmetric constraints of F-type. Target manifolds defined by F-type constraints are always non-compact. In order to obtain nonlinear sigma models on compact manifolds, we have to introduce gauge symmetry to eliminate the degrees of freedom in non-compact directions. All supersymmetric nonlinear sigma models defined on the hermitian symmetric spaces are successfully formulated as gauge theories.

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1 Introduction

Two dimensional (2D) nonlinear sigma models and four dimensional non-abelian gauge theories have several similarities. Both of them enjoy the property of the asymptotic freedom. They are both massless in the perturbation theory, whereas they acquire the mass gap or the string tension in the non-perturbative treatment. Although it is difficult to solve QCD in analytical way, 2D nonlinear sigma models can be solved by the large N expansion and helps us to understand various non-perturbative phenomena in four dimensional gauge theories.

In the nonperturbative treatment of nonlinear σ models, auxiliary field method play an important role. As an example, let us consider the supersymmetric 2D nonlinear σ model with $O(N)$ symmetry. The bosonic field $\vec{A}(x)$ takes the value on the real $N-1$ dimensional sphere S^{N-1} with a radius $\frac{\sqrt{N}}{g}$. The corresponding N fermionic field $\vec{\psi}(x)$ is a Majorana (real) spinor which has the four-fermi type interaction with $O(N)$ symmetry and is called the Gross-Neveu model. The supersymmetric nonlinear sigma model with $O(N)$ symmetry is defined simply as a combination of these two models

$$\mathcal{L}(x) = \frac{1}{2}\{(\partial\vec{A})^2 + \bar{\vec{\psi}} i\partial\!\!\!/ \vec{\psi}\} + \frac{g^2}{8N}(\bar{\vec{\psi}}\vec{\psi})^2$$

where $\vec{A}^2 = \frac{N}{g^2}$ and $\vec{A}\cdot\vec{\psi} = 0$.

This model enjoys three kinds of symmetry: $O(N)$ symmetry, discrete chiral symmetry $\psi \to \gamma_5\psi$ and the supersymmetry which mixes bosonic and fermionic fields. We can find the solution in the large N limit where $N \to \infty$ with g fixed. In order to study the phase structure, it is crucial to introduce auxiliary superfields defined in the superspace with the bosonic coordinate x^μ and the two component fermionic coordinate θ

$$\Phi_0(x,\theta) = A_0(x) + \bar{\theta}\psi_0(x) + \frac{1}{2}\bar{\theta}\theta F_0(x).$$

$A_0(x) \sim \bar{\vec{\psi}}\cdot\vec{\psi}$ describes the scalar bound state of two fermions as the auxiliary field of the Gross-Neveu model. $\psi_0(x) \sim \vec{A}\cdot\vec{\psi}$ describes the fermionic bound state of the fermion and the boson. $F_0(x)$ is the Lagrange multiplier for the constraint: $\vec{A}^2 = \frac{N}{g^2}$

Table 1: **Phase Structure of the $O(N)$ model**

symmetry	order parameter	perturbative	nonperturbative
$O(N)$	$\langle \vec{A} \rangle$	$\times$	$\bigcirc$
chiral symmetry	$\langle A_0 \rangle$	$\bigcirc$	$\times$
supersymmetry	$\langle F_0 \rangle$	$\bigcirc$	$\bigcirc$

of the bosonic nonlinear σ model. These auxiliary fields also play the role of order parameters of various symmetry. With the auxiliary field, the nonlinear lagrangian reduces to a very simple form

$$\mathcal{L}(x) = \frac{1}{2}\{(\partial\vec{A})^2 + \bar{\vec{\psi}} i\partial\!\!\!/\vec{\psi} + \vec{F}^2\} + \Big(2A_0\vec{A}\cdot\vec{F} - A_0\bar{\vec{\psi}}\vec{\psi} + F_0\vec{A}^2 - \bar{\psi}_0\vec{\psi}\vec{A} - \bar{\vec{\psi}}\psi_0\vec{A}\Big) - \frac{N}{g^2}F_0.$$

In the large N field theory, we take into account the dominant vacuum fluctuations of $O(N)$ vector fields $\vec{A}, \vec{\psi}$ and $\vec{F}$ which can be integrated out easily since the lagrangian is simply of quadratic form in these variables. As is shown in the table 1, $O(N)$ symmetry recovers and $\vec{A(x)}$ acquire a mass proportional to $\langle A_0 \rangle$. Chiral symmetry breaks down and fermions have the same mass with bosons.

In this talk, we generalize the auxiliary field formulation to nonlinear σ models with $\mathcal{N} = 2$ supersymmetry in two dimensions, which is equivalent to $\mathcal{N} = 1$ supersymmetry in four dimensions.

2 Nonlinear σ Models in Four Dimensions

When a global symmetry group G breaks down to its subgroup H by a vacuum expectation value $v = \langle \phi \rangle$, there appear massless Nambu-Goldstone (NG) bosons corresponding to broken generators in G/H. At low energies, interactions among these NG bosons are described by nonlinear σ models [1]. In supersymmetric theories, target manifolds of nonlinear σ models must be Kähler manifolds [2, 11]. A

manifold whose metric is given by a Kähler potential $K(\bar{A}, A)$

$$g_{\bar{m}n} = \frac{\partial^2 K(\bar{A}, A)}{\partial \bar{A}^{\bar{m}} \partial A^n}, \tag{1}$$

ia called the Kähler manifold. Since NG bosons must be scalar components of complex chiral superfields

$$\phi(x, \theta) = A(x) + \theta\psi(x) + \frac{1}{2}\theta^2 F(x), \tag{2}$$

there are two possibilities. If the coset G/H itself is a Kähler manifold, both Re $A(x)$ and Im $A(x)$ must be NG-bosons. The effective lagrangian in this case is uniquely determined by the metric of the coset manifold G/H [3, 4, 5, 6]. On the other hand, if the coset G/H is a submanifold of a Kähler manifold, there is at least one chiral superfield whose real or imaginary part is not a NG-boson. This additional massless boson is called the quasi-Nambu-Goldstone(QNG) boson [7, 8]. In this case, the Kähler metric in the direction of QNG boson is not determined by the metric of its subspace G/H, and the effective lagrangian is not unique. In this article, we will confine ourselves to the case of Kähler G/H. The lagrangian of the nonlinear σ model on a Kähler manifold is

$$\begin{aligned} \mathcal{L}(x) \quad &= g_{\bar{m}n}\partial_\mu \bar{A}^{\bar{m}} \partial^\mu A^n + \tfrac{i}{2} g_{\bar{m}n} \left(\bar{\psi}^{\bar{m}} \sigma^\mu (D_\mu \psi)^n + \psi^n \bar{\sigma}^\mu (D_\mu \bar{\psi})^{\bar{m}} \right) \\ &\qquad + \tfrac{1}{4} R_{k\bar{m}l\bar{n}} (\psi^k \psi^l)(\bar{\psi}^{\bar{m}} \bar{\psi}^{\bar{n}}), \end{aligned} \tag{3}$$

where

$$(D_\mu \psi)^n = (\delta^n_m \partial_\mu + \Gamma^n_{lm} \partial_\mu A^l)\psi^m, \qquad \Gamma^n_{lm} = g^{n\bar{k}} \partial_l g_{\bar{k}m}. \tag{4}$$

Once we know the Kähler potential $K(\bar{A}, A)$, we can calculate the metric by (1), the connection by (4) and the lagrangian by (3). This lagrangian is suitable for perturbative calculations. For the nonperturbative study, however, the auxiliary field formulation is hoped for.

3 Auxiliary Field Formulation

Let us start from a known example, the $\mathbf{C}P^{N-1}$ model. Chiral superfield $\phi_i(x, \theta)$ ($i = 1, 2, \cdots, N$) belongs to a fundamental representation of $G = SU(N)$ which is the

isometry of $\mathbf{C}P^{N-1}$. We introduce $U(1)$ gauge symmetry

$$\phi(x,\theta) \longrightarrow \phi'(x,\theta) = e^{i\Lambda(x,\theta)}\phi(x,\theta) \tag{5}$$

to require that $\phi(x,\theta)$ and $\phi'(x,\theta)$ are physically indistinguishable. With a complex chiral superfield, $e^{i\Lambda(x,\theta)}$ is an arbitrary complex number. $U(1)$ gauge symmetry is thus complexified to $U(1)^{\mathbf{C}}$. The identification $\phi \sim \phi'$ defines the complex projective space $\mathbf{C}P^{N-1}$. In order to impose local $U(1)$ gauge symmetry, we have to introduce a $U(1)$ gauge field $V(x,\theta,\bar{\theta})$ with the transformation property $e^V \longrightarrow e^V e^{-i\Lambda+i\Lambda^*}$. Then the lagrangian with a local $U(1)$ gauge symmetry is given by

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}(e^V \vec{\phi}^* \cdot \vec{\phi} - cV) \tag{6}$$

where the last term V is called the Fayet-Iliopoulos D-term. In this model, real scalar superfield $V(x,\theta,\bar{\theta})$ is the auxiliary superfield. The Kähler potential $K(\phi,\phi^*)$ is obtained by eliminating V by using the equation of motion for V

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} K(\phi,\phi^*) = c\int d^2\theta d^2\bar{\theta} \log{(\vec{\phi}^* \cdot \vec{\phi})}. \tag{7}$$

This Kähler potential reduces to the standard Fubini-Study metric of $\mathbf{C}P^{N-1}$

$$K(\phi,\phi^*) = c\log{(1+\sum_{i=1}^{N-1}\phi_i^*\phi_i)} \tag{8}$$

by a choice of gauge fixing

$$\phi_N(x,\theta) = 1. \tag{9}$$

The lagrangian of the $\mathbf{C}P^{N-1}$ model is obtained by substituting the Kähler potential (8) to eqns. (1), (4) and (3). The global symmetry $G = SU(N)$, the isometry of the target space $\mathbf{C}P^{N-1}$, is linearly realized on our ϕ_i fields and our lagrangian (6) with auxiliary field V is manifestly invariant under G. The gauge fixing condition (9) is not invariant under $G = SU(N)$ and we have to perform an appropriate gauge transformation simultaneously to compensate the change of ϕ_N caused by the $SU(N)$ transformation. Therefore the global symmetry $G = SU(N)$ is nonlinearly realized in the gauge fixed theory. In this sense, our lagrangian (6) use the linear realization of G in contrast to the nonlinear lagrangian in terms of the Kähler potential which use the nonlinear realization of G.

Similarly, we introduce M $(M < N)$ copies ϕ_j $(j = 1, 2, \cdots, M)$ of fundamental representations of $SU(N)$ and impose the local gauge symmetry $U(M)$ to identify $\Phi U \sim \Phi$ $(U \in U(M))$ where $\Phi = (\phi_1, \phi_2, \cdots, \phi_M)$ is an $N \times M$ matrix of chiral superfields. Then we obtain the nonlinear σ model on the Grassmann manifold $SU(N)/SU(N-M) \times U(M)$. The lagrangian with an auxiliary $U(M)$ gauge field V reads

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \left(\mathrm{tr} \left(\Phi^\dagger \Phi e^V \right) - c \, \mathrm{tr} V \right) \tag{10}$$

As in the $\mathbf{C}P^{N-1}$ model, the auxiliary field V can be eliminated by the use of its equation motion. We fix the gauge by choosing

$$\Phi = \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix}$$

where φ is an $(N-M) \times M$ matrix valued chiral superfield. Then the Kähler potential reads

$$K(\varphi, \varphi^\dagger) = c \log \det(\mathbf{1}_M + \varphi^\dagger \varphi). \tag{11}$$

Again, the global symmetry $G = SU(N)$ is linearly realized on our fields Φ although the gauge fixed fields φ is no longer a linear realization.

4 Nonlinear Sigma Models with F-term Constraints

The superspace in the four dimensional space-time consists of four bosonic coordinates x^μ $(\mu = 0, 1, 2, 3)$ and a four components Majorana (real) spinor, which is equivalent to a complex Weyl spinor θ^α $(\alpha = 1, 2)$ and its hermitian conjugate $\bar{\theta}$. The chiral superfield defined by (2) depends only on θ but not on $\bar{\theta}$. On the other hand, $\phi^\dagger$ depend on $\bar{\theta}$ but not on θ. If we can make G invariant combinations of chiral superfields, it is possible to introduce in the lagrangian another term called the F-term which can be written as an integral over θ.

Let us try to impose an F-term constraint on $\mathbf{C}P^{N-1}$ model. The simplest constraint consistent with the $U(1)$ gauge symmetry (5) is the quadratic equation

$$\vec{\phi} \cdot \vec{\phi} = 0. \tag{12}$$

With this constraint, the invariance group of the lagrangian is no longer $SU(N)$ symmetry but its subgroup $O(N)$. Any nonvanishing value is forbidden on the righthand side because of the $U(1)$ symmetry. Solution of this F-term constraint: $\vec{\phi}^2 = \vec{x}^2 + (y+iz)\cdot(y-iz) = 0$ written in terms of $\vec{\phi} = (x_i, y, z)$ is given by

$$y - iz = -\frac{x^2}{y+iz} = -\frac{x^2}{\sqrt{2}} \tag{13}$$

where we have chosen a specific gauge $y+iz = \sqrt{2}$. If we have imposed the constraint (12) to $\vec{\phi}$ without introducing the gauge symmetry, the resulting target space would be noncompact Kähler manifold with a QNG boson. The QNG boson is now gauged away by the gauge symmetry (5) as a gauge degree of freedom and has disappeared from the physical spectrum. On substitution of the solution (13) of the constraint to Eq. (7), we obtain the Kähler potential of this model

$$K(x, x^*) = \log\left(1 + \sum_{i=1}^{N-2} x_i^* x_i + \frac{1}{4}\sum_{i,j=1}^{N-2} x_i^{*2} x_j^2\right), \tag{14}$$

which is known as the Kähler potential of the quadratic surface

$$Q^{N-2}(\mathbf{C}) = \frac{SO(N)}{SO(N-2)\times U(1)}.$$

The lagrangian of the model with auxiliary fields is simply obtained from Eq (6) by imposing the F-term constraint (12) with a lagrange multiplier field $\phi_0(x,\theta)$

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}(e^V \vec{\phi}^* \cdot \vec{\phi} - V) + \left(\int d^2\theta \phi_0 \vec{\phi}\cdot\vec{\phi} + \text{h.c.}\right) \tag{15}$$

Let us impose an F-term constraint to the model on the Grassmann manifold $SU(2N)/SU(N)\times U(N)$. Our basic field is the $2N \times N$ matrix valued chiral superfield Φ which transforms linearly under the global symmetry $SU(2N)$. In order to identify Φ with ΦU with $U \in U(N)$, we introduce the $U(N)$ gauge field V. The simplest constraint is again the quadratic constraint $\Phi^T\Phi = 0$. Since this constraint transforms as the symmetric second rank tensor under the gauge group $U(N)$, we introduce the chiral superfield Φ_0 which also transforms as a symmetric second rank tensor under the gauge group $U(N)$. With this constraint, the global symmetry

$SU(2N)$ reduces to its subgroup $SO(2N)$. Thus we obtain the auxiliary field formulation of the $SO(2N)/U(N)$ model

$$\mathcal{L} = \int d^4\theta \left(\mathrm{tr}\left(\Phi^\dagger \Phi e^V\right) - c\,\mathrm{tr}V\right) + \left(\int d^2\theta \;\mathrm{tr}\left(\Phi_0 \Phi^T \Phi\right)\right) + \mathrm{h.c.}\Big). \tag{16}$$

If we insert the simpletic structure

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{1}_N \\ -\mathbf{1}_N & \mathbf{0} \end{pmatrix}. \tag{17}$$

between Φ^T and Φ, the global symmetry reduces to the simpletic group $Sp(N)$. Therefore, the $Sp(N)/U(N)$ model is defined by

$$\mathcal{L} = \int d^4\theta \left(\mathrm{tr}\left(\Phi^\dagger \Phi e^V\right) - c\,\mathrm{tr}V\right) + \left(\int d^2\theta \;\mathrm{tr}\left(\Phi_0 \Phi^T J\Phi\right)\right) + \mathrm{h.c.}\Big). \tag{18}$$

In this case, the chiral auxiliary field Φ_0 transforms as an antisymmetric second rank tensor of the gauge group $U(N)$.

Similarly, we can formulate supersymmetric nonlinear sigma models on hermitian symmetric spaces shown in the table 2 by using auxiliary fields [9]. It should be noted that the third order and the fourth order polynomials appear as the F-term constraints in the case of exceptional groups.

5 Quantum Legendre Transform

If we impose only the symmetry, the nonlinear lagrangian may depend on arbitrary function. For example, consider the simplest $\mathbf{C}P^{N-1}$ model. The following lagrangian with an arbitrary function f is allowed by the global as well as the local symmetry

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}(f(e^V \vec{\phi}^* \cdot \vec{\phi}) - cV). \tag{19}$$

We can prove this arbitrariness disappears and (19) reduces to the simplest lagrangian (6) discussed previously [10]. Namely, we can prove that

$$\int [dV] \exp[i \int d^4\theta (f(e^V \phi^\dagger \phi) - cV)] = \int [dV] \exp[i \int d^4\theta (e^V \phi^\dagger \phi - cV)]$$

Table 2: **Hermitian Symmetric Spaces**

Type	G/H
AIII_1	$\mathbf{C}P^{N-1} = SU(N)/SU(N-1) \times U(1)$
AIII_2	$G_{N,M}(\mathbf{C}) = U(N)/U(N-M) \times U(M)$
BDI	$Q^{N-2}(\mathbf{C}) = SO(N)/SO(N-2) \times U(1)$
CI	$Sp(N)/U(N)$
DIII	$SO(2N)/U(N)$
EIII	$E_6/SO(10) \times U(1)$
EVII	$E_7/E_6 \times U(1)$

First three manifolds, $\mathbf{C}P^{N-1}$, $G_{N,M}(\mathbf{C})$ and $Q^{N-2}(\mathbf{C})$ are called a projective space, a Grassmann manifold and a quadratic surface, respectively.

by using a remarkable property of the quantum Legendre transform in supersymmetric theories which is valid for any vector superfields $\sigma(x,\theta,\bar{\theta})$, $\Phi(x,\theta,\bar{\theta})$

$$\int [d\sigma] \exp\left[i\int d^4xd^4\theta\ (\sigma\Phi - W(\sigma))\right] = \exp\left[i\int d^4xd^4\theta\ U(\Phi)\right], \qquad (20)$$

where $U(\Phi) = \hat{\sigma}(\Phi)\Phi - W(\hat{\sigma}(\Phi))$ is the Legendre transform of W and $\hat{\sigma}$ is the stationary point:

$$\frac{\partial}{\partial\sigma}(\sigma\Phi - W(\sigma))|_{\sigma=\hat{\sigma}} = \Phi - \partial W(\hat{\sigma}) = 0.$$

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